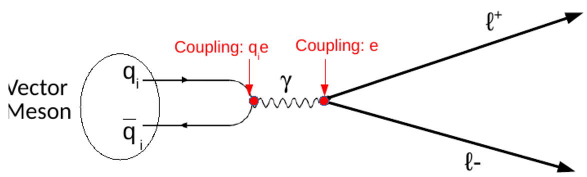
Physics 226: Problem Set #3
Due in Class on Thurs Sept 29, 2015

1. Neutral vector mesons can decay into lepton pairs through $q\overline{q}$ annihiliation via a virtual photon.



This annihilation requires that the wave function of the q and $\overline{q}$ overlap ($\psi(r = 0) \neq 0$), so annihilation is only possible for states with $\ell = 0$. Because the photon has spin $S = 1$ and angular momentum must be conserved at the interaction vertices, the sum of the quark spins must be $S = 1$ (since $\ell = 0$). The lowest lying octet vector meson states have exactly this quark configuration

In general, these decays have small branching ratios, since they are occur via the electromagnetic rather than the strong interaction. We can calculate the partial width for electromagnetic decays of vector mesons in terms of the quark content of the mesons and the $q\overline{q}$ wave function overlap $|\psi(0)|^2$. The result is:

$$\Gamma(V \to \ell^+\ell^-) \propto \frac{\alpha^2 Q^2}{M_V^2}|\psi(0)|^2$$

where α is the fine structure constant and $Q^2 = \left|\sum_i a_i \langle q_i| \mathcal{Q} |q_i\rangle\right|^2$ is the squared sum of the charges of the quarks in the meson (weighted by a_i, the appropriate SU(3) Clebsch-Gordon Coefficients), M_V is the mass of the meson and $\psi(0)$ is the wave function at the origin. The charge operator $\mathcal{Q}$ has the following effect on quark-antiquark pairs:

$$\begin{aligned} \mathcal{Q}\left|u\overline{u}\right\rangle &= \frac{2}{3}\left|u\overline{u}\right\rangle \\ \mathcal{Q}\left|d\overline{d}\right\rangle &= -\frac{1}{3}\left|d\overline{d}\right\rangle \end{aligned}$$

$$\mathcal{Q}\,|s\overline{s}\rangle = \frac{1}{3}\,|s\overline{s}\rangle$$

(a) Using the quark model assignments for the vector mesons

$$\begin{aligned}\rho^0 &= \frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d}) \\ \omega^0 &= \frac{1}{\sqrt{2}}(u\overline{u} + d\overline{d}) \\ \phi^0 &= s\overline{s}\end{aligned}$$

calculate the ratios of the leptonic decay widths for these 3 mesons. For this part of the problem, use SU(3) symmetry to make the approximation that $\psi(0)$ is the same for all 3 states. Use the Particle Data Book to compare your result to the measured values of these ratios.

(b) The ψ is a spin 1 meson with quark content $c\overline{c}$ and mass 3.1 GeV. The c quark has charge 2/3. The $\Upsilon(1S)$, is a spin 1 meson with quark content $b\overline{b}$ and mass 9.460 GeV. The b quark has charge $-1/3$. Because the c-quark and b-quark are so much heavier than the u, d and s, we cannot assume that that these mesons have the same value of $\psi(0)$ as the light vector mesons. The change in $\psi(0)$ with quark mass depends on the radial form of the binding potential. We know, for example, in non-relativistic quantum mechanics that for the hydrogen atom $|\psi(0)|^2 \propto m^3/n^3$ where n is the principle quantum number, while for a potential that is linear in r, $|\psi(0)|^2 \propto m$. We can therefore deduce something about the shape of the potential by studying the quark-mass dependence of the leptonic decay width.

Predict the leptonic decay width for the ψ and the $\Upsilon(1S)$ under each of the following assumptions about the mass dependence of the wave function at the origin:

i. $|\psi(0)|^2$ is independent of M_V,
ii. $|\psi(0)|^2 \propto M_V$
iii. $|\psi(0)|^2 \propto {M_V}^2$.

Use the Particle Data Book to find the true values of these widths. Which of the three predictions agrees best with the data?

2. The lowest lying pseudoscalar and vector multiplets are both SU(3) nonets (octet plus singlet) with no orbital angular momentum. The mass splittings within a multiplet come from the difference in mass between the u, d and s quarks and to a smaller extent from electromagnetic corrections due to the differences in quark charges. Mass splitting between the pseudoscalar and vector multiplets is due to QCD hyperfine splitting. The hyperfine mass splitting for mesons with quark content $q_1\overline{q_2}$ has the form

$$\Delta M(\text{meson}) = A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}|\psi(0)|^2$$

where A is a constant, $\mathbf{S}_i$ is the spin of quark i, m_i is the mass of quark i and $|\psi(0)|$ is the wave function at the origin. For this problem, use $m_u = m_d = 336$ MeV and $m_s = 509$ MeV.

   (a) By comparing the masses of the K^0 and K^{0*} mesons, estimate the value of $A|\psi(0)|^2$

   (b) Use this value of $A|\psi(0)|^2$ and the difference between the up and strange quark masses to predict the splitting between the η and ω. Note: you may assume that the η is a pure SU(3) octet state (this is not completely true, but the mixing with the SU(3) singlet is small). In the case of the ω, the mixing is large. Use the wave function given in the problem above.

3. Consider the process $A + B \to C + D$. The Lorentz invariants that can be defined for this process (in addition to the masses of the A, B, C and D) are the Mandelstam variables:

$$\begin{aligned} s &= (p_A + p_B)^2 \\ t &= (p_A - p_C)^2 \\ u &= (p_A - p_D)^2 \end{aligned}$$

Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$ where m_i is the rest mass of particle i. This tells us that only two the three Mandelstam variables are independent.